Fourier Series, Fourier Integral and Discrete Fourier Transform.

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Introduction.

Fourier Transform is learned centuries and first strict prove is found around year 1829 by Leguen Derehle. For this reason I can be sure that in my text there is exactly nothing that was not proven by somebody else.
The main source of information for me was grate textbook “Base of Math Analysis” by G.M.Fihtengoltz. Else I was using some other sources.
Although the sources I was using give grate explanation, prove for Fourier Transform is given only partial. Math textbook assume that you seek for parts of prove in other topic. There is a lot of material in textbook, therefore search for particular topic can blur attention from the actual problem that must be solved.

1. Orthonormal and Orthogonal sets of functions.

- $f_n(x)$ and $\varphi_n(x)$ are sets of functions defined for x in range [a, b].
- n – index of functions defined in range [c, d].
- for each n in [c, d]
  - $f_n(x)$ and $\varphi_n(x)$ are almost continuous in range [a, b]
  - exist $\lambda_n > 0$

$f_n(x)$ and $\varphi_n(x)$ are called Orthogonal in range [a, b] if and only if

$\int_a^b f_n(x)\varphi_n^*(x)dx = 0$ where both $m\, and\, n \in [c, d]$.

$\varphi_n^*(x)$ is Complex Conjugate of $\varphi_n(x)$.

if $a = a_r + ja_i$ then Complex Conjugate of a is $a^* = a_r - ja_i$.

Notice that if $b = e^{jx}$, then $b^* = e^{-jx}$. This can be proven by Euler's Formula.

Set of functions $f_n(x)$ is called **Pairwise Orthogonal** if each pair of functions $f_n(x)$ and $f_m(x)$ are orthogonal unless $n = m$: $\int_a^b f_n(x)\varphi_n^*(x)dx = \{ \begin{matrix} 0 & n \neq m \\ \lambda_n & n = m \end{matrix} \}$

If $\lambda_n = 1$ for each n, then set of functions $f_n(x)$ is called Orthonormal.

2. Fourier Series.

Let $f(x)$ to be a function defined in range [a, b]

Let $\varphi_n(x)$ to be a set of Pairwise Orthogonal functions in range [a, b], while $n \in [c, d]$.

- c, d and n are integers.
- a and b are real numbers.

Function f(x) for each x in range [c, d] can be approximated by **Fourier Series**.

Fourier Series:

$$f(x) = \sum_{n=c}^{d} F(n)\varphi_n(x)$$

where

$$F(k) = \frac{1}{\lambda_k}\int_a^b f(x)\varphi_k^*(x)\,dx \quad \text{and} \quad \lambda_k = \int_a^b \varphi_k(x)\varphi_k^*\,dx$$

Lets prove equation:

$$f(x) = \sum_{n=c}^{d} F(n)\varphi_n(x)$$

Multiply left and right parts by $\varphi_k^*(x)$ with $k \in [c, d]$.

$$f(x)\varphi_k^*(x) = \sum_{n=c}^{d} F(n)\varphi_n(x)\varphi_k^*(x)$$

Take integral by variable x in range [a, b] from the left and right parts of equation.

$$\int_a^b f(x)\varphi_k^*(x)\,dx = \int_a^b \sum_{n=c}^{d} (F(n)\varphi_n(x)\varphi_k^*(x))\,dx \quad =$$

(by **known** properties of integral)

$$= \quad \sum_{n=c}^{d} F(n)\int_a^b \varphi_n(x)\varphi_k^*(x)\,dx \quad =$$

$$\left(\int_a^b \varphi_n(x)\varphi_k^*(x)\,dx = \left\{\begin{matrix} 0 & k \neq n \\ \lambda_k & k = n \end{matrix}\right\} \textit{because of orthogonality}\right)$$

$$= \quad \lambda_k \cdot F(k) \quad .$$

Therefore

$$F(k) = \frac{1}{\lambda_k}\int_a^b f(x)\varphi_k^*(x)\,dx \quad .$$

That is the searched equation.

3. Fourier Integral.

Let $f(x)$ to be a function defined in range [a, b]

Let $\varphi_n(x)$ to be continuous set of Pairwise Orthogonal functions in range [a, b], while $n \in [c, d]$.

- c, d and n are real numbers since no need to perform summation.
- a and b are real numbers too.

Real function f(x) for each x in range [c, d] can be represented as **Fourier Integral**.

Fourier Integral:

$$f(x) = \int_c^d F(y) \varphi_y(x) dy$$

where

$$F(z) = \frac{1}{\lambda_z} \int_a^b f(x) \varphi_z^*(x) dx \quad \text{and} \quad \lambda_z = \int_a^b \varphi_z(x) \varphi_z^*(x) dx$$

Lets prove equation:

$$f(x) = \int_c^d F(y) \varphi_y(x) dy$$

Multiply left and right part by $\varphi_z^*(x)$, where $\varphi_z^*(x)$ is Complex Conjugate of $\varphi_z(x)$.

$$f(x)\varphi_z^*(x) = (\int_c^d F(y) \varphi_y(x) dy) \varphi_z^*(x)$$

Take integral by x from left and right parts in range [a, b]

$$\int_a^b f(x) \varphi_z^*(x) dx = \int_a^b (\int_c^d F(y) \varphi_y(x) dy) \varphi_z^*(x) dx \quad =$$

(by **known** properties of integral)

$$= \int_c^d F(y) (\int_a^b \varphi_y(x) \varphi_z^*(x) dx) dy \quad =$$

$$(\int_a^b \varphi_y(x) \varphi_z^*(x) dx = \{ \begin{matrix} 0 & y \neq z \\ \lambda_z & y = z \end{matrix} \; because\ of\ orthogonality)$$

$$= \int_c^d F(y) X_z(y) dy \quad (where\ X_z(y) = \{ \begin{matrix} 0 & y \neq z \\ \lambda_z & y = z \end{matrix}) \quad = \quad F(z)\lambda_z$$

The last equality can and will be proven by definition of integral.

$$\int_a^b \varphi_y(x) \varphi_z^*(x) dx = X_z(y) \quad , \quad X_z(y) = \{ \begin{matrix} \lambda_z & z = y \\ 0 & z \neq y \end{matrix}$$

Lets suppose the equality is right.

In this case $F(z)=\frac{1}{\lambda_z}\int_a^b f(x)\varphi_z^*(x)\,dx$.

That is the searched equation.

Lets prove the equality.

If $X_z(y)=\{\begin{matrix} 0 & y\neq z \\ \lambda_z & y=z \end{matrix}$ then $\int_c^d F(y)X_z(y)\,dy=F(z)\lambda_z$

By definition of integral and Riemann Sums

$$\int_c^d t(x)\,dx=\lim_{N\to\infty}\frac{d-c}{N}\sum_{n=0}^{N} t\left(c+n\frac{(d-c)}{N}\right)$$

Therefore

$$\int_c^d F(y)X_z(y)\,dy=\lim_{N\to\infty}\frac{d-c}{N}\sum_{n=0}^{N} F\left(c+n\frac{(d-c)}{N}\right)X_z\left(c+n\frac{(d-c)}{N}\right)$$

Now lets find n, $z=c+n\frac{(d-c)}{N}$.

Particularly it will be right for $n=\frac{(z-c)N}{d-c}$.

This value of n could be not necessary right integer. To ensure that n is indeed integer we can choose N to be multiply of (d-c), i.e. exist integer k, N = k(d-c). Since N is growing to infinity, this assumption is not reducing from generality.

By definition of $X_z(y)$, $X_z(z)=\lambda_z$ and $X_z(y)=0$ for each n,

$$c+n\frac{(d-c)}{N}=y\neq z$$.

Finally, $\int_c^d F(y)X_z(y)\,dy=F(z)\lambda_z$.

Equality proven.

Discrete Fourier Transform.

Select set ϕ of M Pairwise Orthogonal Complex vectors, N elements each.
By definition of orthogonality two vectors φ_m and φ_k of set ϕ are orthogonal iff $\sum_{n=0}^{N-1} \varphi_m[n]\varphi_k^*[n]=0$, where x^* is Complex Conjugate of complex number x .

Sign $\lambda_m=\sum_{n=0}^{N-1} \varphi_m[n]\varphi_m^*[n]$.

Define source vector f with M complex elements.
Exist vector F of N complex elements.

For each $0\le m<M$, $f[m]\in f$, $f[m]=\frac{1}{\lambda_m}\sum_{n=0}^{N-1} F[n]\varphi_m[n]$

and for each $0\le n<N$, $F[n]\in F$, $F[n]=\sum_{m=0}^{M-1} f[m]\varphi_m^*[n]$

$$f[k]=\frac{1}{\lambda_k}\sum_{n=0}^{N-1} F[n]\varphi_k[n]=\frac{1}{\lambda_k}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f[m]\varphi_m^*[n]\varphi_k[n]=$$

$$=\frac{1}{\lambda_k}\sum_{m=0}^{M-1} f[m]\sum_{n=0}^{N-1} \varphi_m^*[n]\varphi_k[n]=\frac{1}{\lambda_k}f[k]\lambda_k=f[k]$$

By this simple computation we prove that:

For any set ϕ of M complex pairwise orthogonal vectors dimension N , if exist $\lambda_m\neq 0$ for each $0\le m<M$, $\varphi_m\in\phi$ and $\lambda_m=\sum_{n=0}^{N-1} \varphi_m[n]\varphi_m^*[n]$, then any complex vector f of dimension M can be recalculated like this:

$$f[m]=\frac{1}{\lambda_m}\sum_{n=0}^{N-1} F[n]\varphi_m[n] \ , \quad 0\le m<M$$

where $F[n]=\sum_{m=0}^{M-1} f[m]\varphi_m^*[n]$, $0\le n<N$.

This two formulas are called **Discrete Fourier Transform** and corresponding **Inverse Discrete Fourier Transform**.

4. Known sets of Pairwise Orthogonal functions.

There are endless possible sets of pairwise orthogonal functions.
For Discrete Fourier Transform it is not enough to define set of pairwise orthogonal continuous functions. We need rather a set of pairwise orthogonal vectors of given dimension.
Define m and n integers, c and r real, $j=\sqrt{-1}$.

$$x \in [-\pi+c, \pi+c]$$
$$n, m\, and\; c \in (-\infty, \infty)$$

This formulas define three most popular sets of pairwise orthogonal vectors of length N:

1. $\varphi_n[m]=e^{j\frac{2\pi}{N}mn}$

This set of pairwise orthogonal vectors consists of no more than N vectors.
This set of vectors have interesting property:

$\varphi_n[N-m]=\varphi_n^*[m]$ and $\varphi_{N-n}[m]=\varphi_n^*[m]$.

2. $\varphi_n[m]=\cos(\frac{2\pi}{N}mn)$

This set of pairwise orthogonal vectors consists of no more than N/2 vectors.
Assume you have more.
Than

$$\cos(\frac{2\pi}{N}m(N-n))=$$
$$=\cos(\frac{2\pi}{N}mN-\frac{2\pi}{N}mn)=$$
$$=\cos(2\pi m)\cos(\frac{2\pi}{N}mn)+\sin(2\pi m)\sin(\frac{2\pi}{N}mn)=$$
$$=\cos(\frac{2\pi}{N}mn)$$

Therefore φ_n and φ_{N-n} are not orthogonal and here from is the limit of N/2 vectors.

3. $\varphi_n[m]=\sin(\frac{2\pi}{N}mn)$

This set of pairwise orthogonal vectors consists of no more than N/2 vectors.
Assume you have more.
Than

$$\sin(\frac{2\pi}{N}m(N-n))=$$
$$=\sin(\frac{2\pi}{N}mN-\frac{2\pi}{N}mn)=$$
$$=\cos(2\pi m)\sin(\frac{2\pi}{N}mn)-\sin(2\pi m)\cos(\frac{2\pi}{N}mn)=$$
$$=\sin(\frac{2\pi}{N}mn)$$

Therefore φ_n and φ_{N-n} are not orthogonal and here from is the limit of N/2 vectors.

Proving known trigonometric equalities.

This equalities are well known and could be proven somewhere else,
however it could be better to have the entire prove chain, at least for less popular formulas.

$\cos(n-m)x-\cos(n+m)x$ =

$(\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b))$

= $(\cos(nx)\cos(mx)+\sin(nx)\sin(mx))-(\cos(nx)\cos(mx)-\sin(nx)\sin(mx))$ =

= $2\sin(nx)\sin(mx)$

$$\boxed{\sin(nx)\sin(mx)=\frac{1}{2}(\cos(n-m)x-\cos(n+m)x)}$$

$\cos(n-m)x+\cos(n+m)x$ =

$(\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b))$

= $(\cos(nx)\cos(mx)+\sin(nx)\sin(mx))+(\cos(nx)\cos(mx)-\sin(nx)\sin(mx))$ =

= $2\cos(nx)\cos(mx)$

$$\boxed{\cos(nx)\cos(mx)=\frac{1}{2}(\cos(n-m)x+\cos(n+m)x)}$$

$\sin(n+m)x-\sin(n-m)x$ =

$(\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b))$

= $(\sin(nx)\cos(mx)+\cos(nx)\sin(mx))-(\sin(nx)\cos(mx)-\cos(nx)\sin(mx))$ =

= $2\cos(nx)\sin(mx)$

$$\boxed{\cos(nx)\sin(mx)=\frac{1}{2}(\sin(n+m)x-\sin(n-m)x)}$$

$\sin(n+m)x+\sin(n-m)x$ =

$(\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b))$

= $(\sin(nx)\cos(mx)+\cos(nx)\sin(mx))+(\sin(nx)\cos(mx)-\cos(nx)\sin(mx))$ =

= $2\sin(nx)\cos(mx)$

$$\boxed{\sin(nx)\cos(mx)=\frac{1}{2}(\sin(n+m)x+\sin(n-m)x)}$$

5. Euler Identities.

Lets take known Taylor-Maclaurin sequences for e^{jx} , e^{-jx} , $\cos(x)$ and $\sin(x)$:

$$e^{jx}=1+jx-\frac{x^2}{2!}-j\frac{x^3}{3!}+\frac{x^4}{4!}+...$$

$$e^{-jx}=1-jx-\frac{x^2}{2!}+j\frac{x^3}{3!}+\frac{x^4}{4!}-...$$

$$\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-...$$

$$\sin(x)=x-j\frac{x^3}{3!}+\frac{x^5}{5!}-...$$

From this sequences we directly derive **Euler Identities.**

Euler Identities:

$$\mathrm{e}^{jx}=\cos(x)+j\sin(x)$$

$$\cos(x)=\frac{\mathrm{e}^{jx}+\mathrm{e}^{-jx}}{2}$$

$$\sin(x)=\frac{\mathrm{e}^{jx}-\mathrm{e}^{-jx}}{2\mathrm{j}}$$

6. Known Integral equalities.

To prove orthogonality of chosen pairs of functions we will need several simple integral equalities.

$$\int_{-\pi+c}^{\pi+c} dx = \pi + c - (-\pi + c) = 2\pi$$

or simple $\boxed{\int_{-\pi+c}^{\pi+c} dx = 2\pi}$

For real a, $a \neq 0$:

$$\int_{-\pi+c}^{\pi+c} \cos(ax)dx = \sin\frac{(ax)}{a}\Big|_{-\pi+c}^{\pi+c} =$$

$$= \frac{1}{a}(\sin(a(\pi+c)) - \sin(a(c-\pi))) =$$

$$= \frac{1}{a}((\sin(a\pi)\cos(ac) + \sin(ac)\cos(a\pi)) - (\sin(ac)\cos(a\pi) - \sin(a\pi)\cos(ac))) =$$

$$= 0 \cdot \frac{1}{a} = 0$$

or simple $\boxed{\int_{-\pi+c}^{\pi+c} \cos(ax)dx = 0}$

$$\int_{-\pi+c}^{\pi+c} \sin(ax)\,dx = \frac{-\cos(ax)}{a}\Big|_{-\pi+c}^{\pi+c} =$$

$$= -\frac{1}{a}(\cos(a(\pi+c)) - \cos(a(-\pi+c))) =$$

$$= \frac{1}{a}(\cos(a(c-\pi)) - \cos(a(\pi+c))) =$$

$$= \frac{1}{a}((\cos(ac)\cos(a\pi) + \sin(ac)\sin(a\pi)) - (\cos(ac)\cos(a\pi) - \sin(ac)\sin(a\pi))) =$$

$$= \frac{1}{a}(2\sin(ac)\sin(a\pi)) = 0$$

therefore $\boxed{\int_{-\pi+c}^{\pi+c} \sin(ax)dx = 0}$ for any a, including 0.

7. Prove of continuous orthogonality.

Lets start to prove orthogonality for continuous **trigonometric** functions:

$$\int_{-\pi+c}^{\pi+c} \sin(nx)\sin(mx)\,dx \;=\; \left(\sin(nx)\sin(mx)=\frac{1}{2}(\cos(n-m)x-\cos(n+m)x)\right)$$

$$= \frac{1}{2}\left(\int_{-\pi+c}^{\pi+c} \cos(n-m)x\,dx - \int_{-\pi+c}^{\pi+c} \cos(n+m)x\,dx\right) =$$

$$\left(\int_{-\pi+c}^{\pi+c} \cos(ax)dx = \begin{cases} 0 & a\neq 0 \\ 2\pi & a=0 \end{cases}\right) = \begin{cases} 0 & n\neq m \\ \pi & n=m \end{cases}$$

$$\boxed{\int_{-\pi+c}^{\pi+c} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & n\neq m \\ \pi & n=m \end{cases}}$$

$$\int_{-\pi+c}^{\pi+c} \cos(nx)\cos(mx)\,dx \;=\; \left(\cos(nx)\cos(mx)=\frac{1}{2}(\cos(n-m)x+\cos(n+m)x)\right)$$

$$= \frac{1}{2}\left(\int_{-\pi+c}^{\pi+c} \cos(n-m)x\,dx + \int_{-\pi+c}^{\pi+c} \cos(n+m)x\,dx\right) =$$

$$\left(\int_{-\pi+c}^{\pi+c} \cos(ax)dx = \begin{cases} 0 & a\neq 0 \\ 2\pi & a=0 \end{cases}\right) = \begin{cases} 0 & n\neq m \\ \pi & n=m \end{cases}$$

$$\boxed{\int_{-\pi+c}^{\pi+c} \cos(nx)\cos(mx)\,dx = \begin{cases} 0 & n\neq m \\ \pi & n=m \end{cases}}$$

This two equalities prove that:

- $\sin(nx)$ is a set of Pairwise Orthogonal functions.
- $\cos(nx)$ is a set of Pairwise Orthogonal functions.

The last set of Pairwise Orthogonal functions to prove is for z-Transform.

$$\int_{-\pi+c}^{\pi+c} (r e^{jx})^n (r e^{jx})^{-m}\,dx \;=\; r^{n-m}\int_{-\pi+c}^{\pi+c} e^{j(n-m)x}\,dx \;=\; \left(e^{jax}=\cos(ax)+j\sin(ax)\right)$$

$$= r^{n-m}\left(\int_{-\pi+c}^{\pi+c} \cos(n-m)x\,dx + j\int_{-\pi+c}^{\pi+c} \sin(n-m)x\,dx\right) = \begin{cases} 0 & n\neq m \\ 2\pi & n=m \end{cases}$$

$$\boxed{\int_{-\pi+c}^{\pi+c} (r e^{jx})^n (r e^{jx})^{-m}\,dx = \begin{cases} 0 & n\neq m \\ 2\pi & n=m \end{cases}}$$

Prove discrete orthogonality of vectors.

I will start from inductive prove of three useful theorems:

Theorem 1: For any integer number N and $0 \le m < N$

$$\sum_{n=0}^{N-1} e^{j2\pi \frac{mn}{N}} = \{ \begin{matrix} 0 & for & m\%N \neq 0 \\ N & for & m\%N = 0 \end{matrix}$$

Theorem 2: For any even number N and $0 \le m < \frac{N}{2}$

$$\sum_{n=0}^{N-1} \cos(2\pi \frac{mn}{N}) = \{ \begin{matrix} 0 & for & m\%N \neq 0 \\ N & for & m\%N = 0 \end{matrix}$$

Theorem 3: For any even number N and $0 \le m < \frac{N}{2}$

$$\sum_{n=0}^{N-1} \sin(2\pi \frac{mn}{N}) = 0$$

Prove Theorem 1: Lets start induction on N .

$N=2$:

$$\sum_{n=0}^{N-1} e^{j2\pi \frac{mn}{N}} = e^{j2\pi \frac{m\cdot 0}{2}} + e^{j2\pi \frac{m\cdot 1}{2}} = 1 + e^{j\pi m} =$$

$$(e^{j\pi m} = \cos(\pi m) + jsin(\pi m) = \cos(\pi m) = (-1)^m)$$

$$= 1 + (-1)^m = \{ \begin{matrix} 0 & for & m\%2 \neq 0 \\ 2 & for & m\%2 = 0 \end{matrix}$$

$N=4$:

$$S_4 = e^{j2\pi \frac{m\cdot 0}{4}} + e^{j2\pi \frac{m\cdot 1}{4}} + e^{j2\pi \frac{m\cdot 2}{4}} + e^{j2\pi \frac{m\cdot 3}{4}}$$

$$e^{j2\pi \frac{m\cdot 0}{4}} = 1$$

$$e^{j2\pi \frac{m\cdot 1}{4}} = e^{j\frac{\pi}{2}m} = (\cos(\frac{\pi}{2}) + jsin(\frac{\pi}{2}))^m = (0 + j)^m = j^m$$

$$e^{j2\pi \frac{m\cdot 2}{4}} = e^{j\pi m} = (\cos(\pi) + jsin(\pi))^m = (-1)^m$$

$$e^{j2\pi \frac{m\cdot 3}{4}} = e^{j(\pi + \frac{\pi}{2})m} = (\cos(\pi + \frac{\pi}{2}) + jsin(\pi + \frac{\pi}{2}))^m = (-j)^m$$

$$S_4 = 1 + j^m + (-1)^m + (-j)^m$$

Split calculation to the four cases.

Case 1: $m\%4=0$

$(-1)^m=1$

$j^m=(j\cdot j)\cdot(j\cdot j)=(-1)\cdot(-1)=1$

$(-j)^m=(-j\cdot -j)\cdot(-j\cdot -j)=1$

$s_4=1+1+1+1=4$

Case 2: $m\%4=1$

$(-1)^m=-1$

$j^m=j$

$(-j)^m=-j$

$S_4=1-1+j-j$

Case 3: $m\%4=2$

$(-1)^m=-1\cdot -1=1$

$j^m=j\cdot j=-1$

$(-j)^m=-j\cdot -j=-1$

$S_4=1+1-1-1=0$

Case 4: $m\%4=3$

$(-1)^m=-1\cdot -1\cdot -1=-1$

$j^m=j\cdot j\cdot j=-j$

$(-j)^m=-j\cdot -j\cdot -j=j$

$S_4=1-1-j+j=0$

Assume by induction $\sum_{n=0}^{\frac{N}{2}-1} e^{j2\pi\frac{mn}{N/2}}=0$ and $m\%(N/2)\neq 0$.

I want to prove that $\sum_{n=0}^{N-1} e^{j2\pi\frac{mn}{N}}=0$.

Split the sum to odd and even summators.

$$\sum_{n=0}^{N-1} e^{j2\pi\frac{mn}{N}}=\sum_{n=0}^{\frac{N}{2}-1} e^{j2\pi\frac{m(2n)}{N}}+\sum_{n=0}^{\frac{N}{2}-1} e^{j2\pi\frac{m(2n+1)}{N}}=$$

$$=\sum_{n=0}^{\frac{N}{2}-1} e^{j2\pi\frac{mn}{N/2}}+e^{j2\pi\frac{m}{N}}\cdot\sum_{n=0}^{\frac{N}{2}-1} e^{j2\pi\frac{mn}{N/2}}=$$

$$\left(\sum_{n=0}^{\frac{N}{2}-1} e^{j2\pi\frac{mn}{N/2}}=0\, as\ assumed\right)$$

$$=0+e^{j2\pi\frac{m}{N}}\cdot 0=0$$

Assume by induction $\sum_{n=0}^{\frac{N}{2}-1} e^{j2\pi\frac{mn}{N/2}}=\frac{N}{2}$ and $m\%(N/2)=0$.

$$\sum_{n=0}^{N-1} e^{j2\pi\frac{mn}{N}}=\sum_{n=0}^{\frac{N}{2}-1} e^{j2\pi\frac{mn}{N/2}}+e^{j2\pi\frac{m}{N}}\cdot\sum_{n=0}^{\frac{N}{2}-1} e^{j2\pi\frac{mn}{N/2}}=$$

$$=\frac{N}{2}+e^{j2\pi\frac{m}{N}}\cdot\frac{N}{2}=\begin{cases}0 & m\%N\neq 0\\ N & m\%N=0\end{cases}$$

$$e^{j2\pi\frac{m}{N}}=\cos(2\pi\frac{m}{N})+j\sin(2\pi\frac{m}{N})\ ;$$

assume $m\%N=0$: $\cos(2\pi\frac{m}{N})=1$, $\sin(2\pi\frac{m}{N})=0$;

$$e^{j2\pi\frac{m}{N}}=1\rightarrow\frac{N}{2}+\frac{N}{2}\cdot e^{j2\pi\frac{m}{N}}=\frac{N}{2}+\frac{N}{2}=N$$

assume $m\%N\neq 0$:

known $m\%N/2=0\rightarrow m\%N=N/2$; $\cos(2\pi\frac{m}{N})=-1$, $\sin(2\pi\frac{m}{N})=0$;

$$e^{j2\pi\frac{m}{N}}=-1\rightarrow\frac{N}{2}-\frac{N}{2}\cdot e^{j2\pi\frac{m}{N}}=\frac{N}{2}-\frac{N}{2}=0$$

From the above calculations we get an exact prove for “**Theorem 1**” that is

$$\sum_{n=0}^{N-1} e^{j2\pi\frac{mn}{N}} = \{ \begin{matrix} 0 & for & m\%N \neq 0 \\ N & for & m\%N = 0 \end{matrix} \quad .$$

Prove Theorem 2 and 3:

From the Theorem 1 $\sum_{n=0}^{N-1} e^{j2\pi\frac{mn}{N}} = \{ \begin{matrix} 0 & for & m\%N \neq 0 \\ N & for & m\%N = 0 \end{matrix}$.

Remind Euler formula: $e^{jx} = \cos(x) + jsin(x)$.

$$\sum_{n=0}^{N-1} e^{j2\pi\frac{mn}{N}} = \sum_{n=0}^{N-1} \cos(2\pi\frac{mn}{N}) + j\cdot\sum_{n=0}^{N-1} \sin(2\pi\frac{mn}{N})$$

Therefore

$$\sum_{n=0}^{N-1} \cos(2\pi\frac{mn}{N}) = \{ \begin{matrix} 0 & m\%N \neq 0 \\ N & m\%N = 0 \end{matrix}$$

$\sum_{n=0}^{N-1} \sin(2\pi\frac{mn}{N}) = 0$ for any $0 \leq m < \frac{N}{2}$.

By this Theorems 2 and 3 are proven.

By proven theorems it is easy to get orthogonality of vector sets.
Select two values q and p : $0 \le q < N$, $0 \le p < N$.
That mean $-N < q-p < N$.
In this condition $(q-p)\% N = 0 \rightarrow q-p=0$.
Write the **Theorem 1** equation while $m=q-p$.

$$\sum_{n=0}^{N-1} e^{j2\pi\frac{(q-p)n}{N}} = \{ \begin{matrix} 0 & for & q-p \ne 0 \\ N & for & q-p=0 \end{matrix}$$

$$\boxed{\sum_{n=0}^{N-1} e^{j2\pi\frac{qn}{N}} e^{-j2\pi\frac{pn}{N}} = \{ \begin{matrix} 0 & for & q \ne p \\ N & for & q=p \end{matrix}}$$

The last equation is exactly the definition of orthogonality.
Conclusion: set of vectors defined by $e^{j2\pi\frac{nm}{N}}$ is a set of pairwise orthogonal vectors for any integer N and $0 \le n < N$, $0 \le m < N$.
Similar for Theorem 2 while $m=q-p$.

$$\sum_{n=0}^{N-1} \cos(2\pi\frac{(q-p)n}{N}) = \{ \begin{matrix} 0 & for & q-p \ne 0 \\ N & for & q-p=0 \end{matrix} .$$

Remind equation $\cos(nx)\cos(mx) = \frac{1}{2}(\cos(n-m)x + \cos(n+m)x)$.

By this equation the above sum can be rewritten:

$$\sum_{n=0}^{N-1} \cos(2\pi\frac{pn}{N})\cos(2\pi\frac{qn}{N}) =$$

$$= \frac{1}{2}(\sum_{n=0}^{N-1} \cos(2\pi\frac{(p-q)n}{N}) + \sum_{n=0}^{N-1} \cos(2\pi\frac{(p+q)n}{N}))$$

$0 \le p < N$, $0 \le q < N$ $\rightarrow$ therefore $(p+q)\% N = 0 \rightarrow p=q=0$.
If $(p-q)\% N = 0$ and $p \ne 0$ then:

$$\sum_{n=0}^{N-1} \cos(2\pi\frac{(p+q)n}{N}) = 0 \ ; \ \sum_{n=0}^{N-1} \cos(2\pi\frac{(p-q)n}{N}) = \sum_{n=0}^{N-1} \cos(0) = N$$

$$\sum_{n=0}^{N-1} \cos(2\pi\frac{pn}{N})\cos(2\pi\frac{qn}{N}) = \frac{N}{2} .$$

else, if $p=q=0$ then:

$$\sum_{n=0}^{N-1} \cos(2\pi \frac{(p+q)n}{N}) = \sum_{n=0}^{N-1} \cos(2\pi \frac{(p-q)n}{N}) = \sum_{n=0}^{N-1} \cos(0) = N$$

$$\sum_{n=0}^{N-1} \cos(2\pi \frac{pn}{N}) \cos(2\pi \frac{qn}{N}) = N$$

else, if $p \neq q$ then:

$$\sum_{n=0}^{N-1} \cos(2\pi \frac{(p+q)n}{N}) = \sum_{n=0}^{N-1} \cos(2\pi \frac{(p-q)n}{N}) = 0$$

$$\sum_{n=0}^{N-1} \cos(2\pi \frac{pn}{N}) \cos(2\pi \frac{qn}{N}) = 0 \quad .$$

Conclusion: Set of vectors, defined by function $f_p[n] = \cos(2\pi \frac{pn}{N})$

is Pairwise Orthogonal in range $[0, \frac{N}{2})$:

$$\boxed{\sum_{n=0}^{N-1} \cos(2\pi \frac{pn}{N}) \cos(2\pi \frac{qn}{N}) = \{ \begin{array}{cc} 0 & p \neq q \\ \frac{N}{2} & p=q \neq 0 \\ N & p=q=0 \end{array}}$$

Similar computation can be done for set of vectors defined by $f_p[n] = \sin(2\pi \frac{pn}{N})$

in range $[0, \frac{N}{2})$. This time I will use equation $\sin(nx)\sin(mx) = \frac{1}{2}(\cos(n-m)x - \cos(n+m)x)$.

$$\sum_{n=0}^{N-1} \sin(2\pi \frac{pn}{N}) \sin(2\pi \frac{qn}{N}) =$$

$$= \frac{1}{2}(\sum_{n=0}^{N-1} \cos(2\pi \frac{(p-q)n}{N}) - \sum_{n=0}^{N-1} \cos(2\pi \frac{(p+q)n}{N}))$$

The calculations are exactly the same as it was for case of cosine.

Conclusion: Set of vectors, defined by function $f_p[n] = \sin(2\pi \frac{pn}{N})$

is Pairwise Orthogonal in range $[0, \frac{N}{2})$:

$$\boxed{\sum_{n=0}^{N-1} \sin(2\pi \frac{pn}{N}) \sin(2\pi \frac{qn}{N}) = \{ \begin{array}{cc} 0 & p \neq q \\ \frac{N}{2} & p=q \neq 0 \\ 0 & p=q=0 \end{array}}$$

Known Discrete Fourier Transforms.

Given the proven above sets of pairwise orthogonal vectors, we can define Discrete Fourier Transform with different basis functions.

1.
$$\sum_{n=0}^{N-1} e^{j2\pi\frac{qn}{N}} e^{-j2\pi\frac{pn}{N}} = \{ \begin{matrix} 0 & for & q \neq p \\ N & for & q = p \end{matrix}$$

By general definition of Discrete Fourier Transform

$$f[m] = \frac{1}{\lambda_m} \sum_{n=0}^{N-1} F[n] \varphi_m[n] \quad , \quad 0 \leq m < M$$

where $F[n] = \sum_{m=0}^{M-1} f[m] \varphi_m^*[n] \quad , \quad 0 \leq n < N$.

In this case $N = M$, therefore

$f[m] = \frac{1}{N} \sum_{n=0}^{N-1} F[n] e^{j2\pi\frac{mn}{N}}$, where $0 \leq m < N$, is called

Complex Sinusoid Discrete Fourier Transform.

$F[n] = \sum_{m=0}^{M-1} f[m] e^{-j2\pi\frac{mn}{N}}$, where $0 \leq n < N$, is called

Inverse Complex Sinusoid Discrete Fourier Transform.

2.

$$\sum_{n=0}^{N-1} \cos(2\pi \frac{pn}{N}) \cos(2\pi \frac{qn}{N}) = \{ \begin{matrix} 0 & p \neq q \\ \frac{N}{2} & p = q \neq 0 \\ N & p = q = 0 \end{matrix}$$

where $0 \leq p - q < \frac{N}{2}$.

By general definition of Discrete Fourier Transform

$$f[m] = \frac{1}{\lambda_m} \sum_{n=0}^{N-1} F[n] \varphi_m[n] \quad , \quad 0 \leq m < M$$

where $F[n] = \sum_{m=0}^{M-1} f[m] \varphi_m^*[n] \quad , \quad 0 \leq n < N$.

In this case $M = \frac{N}{2}$.

$$f[m] = \frac{1}{N} \sum_{n=0}^{N-1} F[n] \cos(2\pi \cdot 0) = \frac{1}{N} \sum_{n=0}^{N-1} F[n] \text{ for } m=0 \text{ .}$$

$$f[m] = \frac{2}{N} \sum_{n=0}^{N-1} F[n] \cos(2\pi \frac{mn}{N}) \text{ for } m>0 \text{ .}$$

$$F[n] = \sum_{m=0}^{M-1} f[m] \cos(2\pi \frac{mn}{N}) \text{ .}$$

However, we know that

$$\cos(2\pi \frac{m(N-n)}{N}) = \cos(2\pi \frac{mN}{N} - 2\pi \frac{mn}{N}) =$$
$$= \cos(2\pi m) \cos(2\pi \frac{mn}{N}) + \sin(2\pi m) \sin(2\pi \frac{mn}{N}) =$$
$$= \cos(2\pi \frac{mn}{N})$$

Therefore $F[N-n] = \sum_{m=0}^{M-1} f[m] \cos(2\pi \frac{m(N-n)}{N}) = F[n]$.

Here from, for any $m>0$,

$$f[m]=\frac{2}{N}\sum_{n=0}^{N-1}F[n]\cos(2\pi\frac{mn}{N})=$$

$$=\frac{2}{N}(\sum_{n=0}^{\frac{N}{2}-1}F[n]\cos(2\pi\frac{mn}{N})+\sum_{n=0}^{\frac{N}{2}-1}F[N-n]\cos(2\pi\frac{m(N-n)}{N}))=$$

$$=\frac{2}{N/2}\sum_{n=0}^{\frac{N}{2}-1}F[n]\cos(\pi\frac{mn}{N/2})$$

Remind $M=\frac{N}{2}$, therefore $f[m]=\frac{2}{M}\sum_{n=0}^{M-1}F[n]\cos(\pi\frac{mn}{M})$ for $m>0$.

for $m=0$:

$$f[m]=\frac{1}{N}\sum_{n=0}^{N-1}F[n]=\frac{1}{N}(\sum_{n=0}^{\frac{N}{2}-1}F[n]+\sum_{n=0}^{\frac{N}{2}-1}F[N-n])=\frac{1}{N/2}\sum_{n=0}^{\frac{N}{2}-1}F[n]$$

or $f[m]=\frac{1}{M}\sum_{n=0}^{M-1}F[n]$.

Writing another way: $f[m]=\sigma_m\sum_{n=0}^{M-1}F[n]\cos(\pi\frac{mn}{M})$, where $\sigma_m=\{\begin{matrix}\frac{1}{M} & m=0\\ \frac{2}{M} & m\neq 0\end{matrix}$

This formula is called **Cosine Discrete Fourier Transform**.

Corresponding inverse transform is recalculated like this:

$$F[n]=\sum_{m=0}^{M-1}f[m]\cos(2\pi\frac{mn}{N})=\sum_{m=0}^{M-1}f[m]\cos(\pi\frac{mn}{M}) \quad .$$

Finally, formula $F[n]=\sum_{m=0}^{M-1}f[m]\cos(\pi\frac{mn}{M})$ is called **Inverse Cosine Discrete Fourier Transform**.

3.

$$\sum_{n=0}^{N-1} \sin(2\pi \frac{pn}{N}) \sin(2\pi \frac{qn}{N}) = \{ \begin{matrix} 0 & p \neq q \\ \frac{N}{2} & p = q \neq 0 \\ 0 & p = q = 0 \end{matrix}$$

By general definition of Discrete Fourier Transform

$$f[m] = \frac{1}{\lambda_m} \sum_{n=0}^{N-1} F[n] \varphi_m[n] \quad , \quad 0 \leq m < M$$

where $F[n] = \sum_{m=0}^{M-1} f[m] \varphi_m^*[n] \quad , \quad 0 \leq n < N$.

For $m=0$, obviously $f[m]=0$.

For $m>0$, $f[m] = \frac{1}{N/2} \sum_{n=0}^{N-1} F[n] \sin(2\pi \frac{mn}{N})$.

Corresponding $F[n] = \sum_{m=0}^{\frac{N}{2}-1} f[m] \sin(2\pi \frac{mn}{N})$.

Remind that

$$\sin(2\pi \frac{m(N-n)}{N}) = \sin(2\pi \frac{mN}{N} - 2\pi \frac{mn}{N}) =$$
$$= \sin(2\pi m)\cos(2\pi \frac{mn}{N}) - \cos(2\pi m)\sin(2\pi \frac{mn}{N}) =$$
$$= \sin(2\pi \frac{mn}{N})$$

Therefore $F[N-n] = F[n]$.

$$f[m] = \frac{1}{N/2} \sum_{n=0}^{N-1} F[n] \sin(2\pi \frac{mn}{N}) =$$

Here from

$$= \frac{1}{N/2} \left(\sum_{n=0}^{\frac{N}{2}-1} F[n] \sin(2\pi \frac{mn}{N}) + \sum_{n=0}^{\frac{N}{2}-1} F[n] \sin(2\pi \frac{m N-n}{N}) \right)$$

$$= \frac{2}{N/2} \sum_{n=0}^{\frac{N}{2}-1} F[n] \sin(2\pi \frac{mn}{N})$$

Finally, formula $f[m] = \frac{2}{M} \sum_{n=0}^{M-1} F[n] \sin(\pi \frac{mn}{M})$ is called **Sine Discrete Fourier Transform**.

Similarly, inverse calculation:

$$F[n] = \sum_{m=0}^{\frac{N}{2}-1} f[m] \sin(2\pi \frac{mn}{N}) = \sum_{m=0}^{M-1} f[m] \sin(\pi \frac{mn}{M}) \quad .$$

Formula $F[n] = \sum_{m=0}^{M-1} f[m] \sin(\pi \frac{mn}{M})$ is called **Inverse Sine Discrete Fourier Transform**.

8. Known formulas, proven in previous topics.

Given the proven sets of pairwise orthogonal functions, we can state this known cases of **Continuous Fourier Series**:

Sine Series:

$$f(x)=\sum_{n=-\infty}^{\infty} F(n)\sin(nx)\,dx$$

$$F(k)=\frac{1}{\pi}\int_{-\pi+c}^{\pi+c} f(x)\sin(kx)\,dx$$

Cosine Series:

$$f(x)=\sum_{n=-\infty}^{\infty} F(n)\cos(nx)\,dx$$

$$F(k)=\frac{1}{\pi}\int_{-\pi+c}^{\pi+c} f(x)\cos(kx)\,dx$$

z-Transform:

$$f(x)=\sum_{n=-\infty}^{\infty} F(n)(r\,e^{jx})^{n}\,dx$$

$$F(k)=\frac{1}{\pi}\int_{-\pi+c}^{\pi+c} f(x)(r\,e^{jx})^{-k}\,dx$$

And the last transform is z-Transform with r = 1, else called Complex Sinusoid Transform, since $y=e^{jx}$ is a formula of Complex Sinusoid.

Complex Sinusoid Series:

$$f(x)=\sum_{n=-\infty}^{\infty} F(n)e^{jnx}\,dx$$

$$F(k)=\frac{1}{\pi}\int_{-\pi+c}^{\pi+c} f(x)e^{-jnx}\,dx$$

As a natural extension, we can state this known cases of **Continuous Fourier Integrals**:

Sine Fourier Integral:

$$f(x)=\int_{-\pi+c}^{\pi+c} F(n)\sin(nx)\,dx$$

$$F(k)=\frac{1}{\pi}\int_{-\pi+c}^{\pi+c} f(x)\sin(kx)\,dx$$

Cosine Fourier Integral:

$$f(x)=\int_{-\pi+c}^{\pi+c} F(n)\cos(nx)\,dx$$

$$F(k)=\frac{1}{\pi}\int_{-\pi+c}^{\pi+c} f(x)\cos(kx)\,dx$$

z-Transform:

$$f(x)=\int_{-\pi+c}^{\pi+c} F(n)(r\,e^{jx})^{n}\,dx$$

$$F(k)=\frac{1}{\pi}\int_{-\pi+c}^{\pi+c} f(x)(r\,e^{jx})^{-k}\,dx$$

Complex Sinusoid Fourier Integral:

$$f(x)=\int_{-\pi+c}^{\pi+c} F(n)\,e^{jnx}\,dx$$

$$F(k)=\frac{1}{\pi}\int_{-\pi+c}^{\pi+c} f(x)e^{-jnx}\,dx$$

And finally, we can state this known cases of **Discrete Fourier Transforms**:

Sine Discrete Fourier Transform:

$$f[m]=\frac{2}{N}\sum_{n=0}^{N-1} F[n]\sin(\pi\frac{mn}{N})$$

$$F[n]=\sum_{m=0}^{N-1} f[m]\sin(\pi\frac{mn}{N})$$

Cosine Discrete Fourier Transform:

$$f[m]=\sigma_m\sum_{n=0}^{N-1} F[n]\cos(\pi\frac{mn}{N})$$, where $\sigma_m=\{\begin{matrix}\frac{1}{N} & m=0\\ \frac{2}{N} & m\neq 0\end{matrix}$

$$F[n]=\sum_{m=0}^{N-1} f[m]\cos(\pi\frac{mn}{N})$$

Complex Sinusoid Discrete Fourier Transform:

$$f[m]=\frac{1}{N}\sum_{n=0}^{N-1} F[n]e^{j2\pi\frac{mn}{N}}$$

$$F[n]=\sum_{m=0}^{M-1} f[m]e^{-j2\pi\frac{mn}{N}}$$

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