Orthogonality of complex continuous functions.

Author: Arkadi Kagan.
arkadi_kagan@hotmail.com

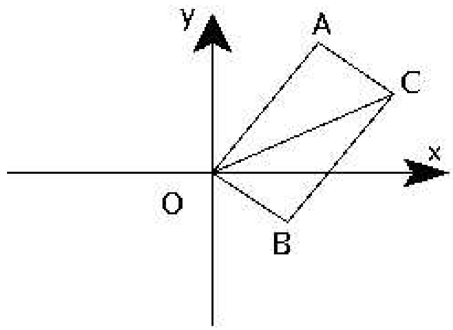
Introduction.

In this document I will try to prove validity of orthogonality law for complex continious functions. This law is known centuries and most compact is referred like this:

Two functions φ and ψ are orthogonal iff $(\varphi, \psi) = 0$.

The prove I am starting from geometric definition of orthogonality and extand the definition step by step.

Orthogonality of vectors is known as geometric idea in two and three-dimensional Euclidean Spaces.
In **two-dimensional space**, lets take two orthogonal vectors A and B.
Remind geometric law for summation of two vectors:



Vector C is defined to be a sum of vectors A and B.
From simple geometric calculations, Euclidean coordinates of vector C are:

$$x_c = x_a + x_b \quad , \quad y_c = y_a + y_b$$

where

$$A = (x_a , y_a) \quad , \quad B = (x_b , y_b) \quad , \quad C = (x_c , y_c) \quad .$$

Recall Pythagorean Theorem:
Two vectors a and b are orthogonal and $c = a\,(+)\,b$ iff $|c|^2 = |a|^2 + |b|^2$.
Where $(+)$ is geometric summation of vectors and $|a|$ is Euclidean length of vector, defined by $|a|^2 = x_a^2 + y_a^2$ for two-dimensional vector a or $|a|^2 = x_a^2 + y_a^2 + z_a^2$ in three-dimensional space.

Theorem 1: If A and B are orthogonal, then $x_a x_b + y_a y_b = 0$.
Prove: Lets compute vector lengths of vectors A, B and C (squares of lengths).

$$|A|^2 = x_a^2 + y_a^2 \quad , \quad |B|^2 = x_b^2 + y_b^2 \quad , \quad |C|^2 = x_c^2 + y_c^2$$

and by Pythagorean Theorem $|C|^2 = |A|^2 + |B|^2$.
Lets write full computation of $|C|^2$ in terms of A and B coordinates:

$$|C|^2 = (x_a + x_b)^2 + (y_a + y_b)^2 \quad \text{or} \quad |C|^2 = (x_a^2 + y_a^2) + (x_b^2 + y_b^2) \quad .$$

$$x_a^2 + 2x_a x_b + x_b^2 + y_a^2 + 2y_a y_b + y_b^2 = x_a^2 + x_b^2 + y_a^2 + y_b^2 \quad ;$$

$$2x_a x_b + 2y_a y_b = 0 \quad ;$$

$x_a x_b + y_a y_b = 0$ that is what was required to be proven.

Theorem 2: If $x_a x_b + y_a y_b = 0$ then A and B are orthogonal vectors.
Prove: If $x_a x_b + y_a y_b = 0$, then $|C|^2 = |A|^2 + |B|^2$ - symmetric to prove of Theorem 1. By Pythagorean Theorem $|C|^2 = |A|^2 + |B|^2$, therefore vectors A and B are orthogonal.

In **three-dimensional space** the things are quit similar.

As known from stereometric laws, two vectors A and B are orthogonal iff exist plane P:

- The plane P is defined by three points (A, B, O), where O is coordinates system centre.
- Vectors A and B are orthogonal on two-dimensional space in plane P.

Define vector C that is a geometric summation of vectors A and B in plane P. By definition of stereometric vector length and Pythagorean theorem, we can compute lengths of A, B and C like this:

$|A|^2=x_a^2+y_a^2+z_a^2$, $|B|^2=x_b^2+y_b^2+z_b^2$, $|C|^2=x_c^2+y_c^2+z_c^2$ and else

$|C|^2=|A|^2+|B|^2$.

Coordinates of vector C are computed like this:

$x_c=x_a+x_b$, $y_c=y_a+y_b$, $z_c=z_a+z_b$.

Where $A=(x_a, y_a, z_a)$, $B=(x_b, y_b, z_b)$, $C=(x_c, y_c, z_c)$.

By simple substitution equation for $|C|^2$ can be rewrote like this:

$|C|^2=(x_a+x_b)^2+(y_a+y_b)^2+(z_a+z_b)^2$ and else $|C|^2=(x_a^2+y_a^2+z_a^2)+(x_b^2+y_b^2+z_b^2)$.

$$x_a^2+2x_a x_b+x_b^2+y_a^2+2y_a y_b+y_b^2+z_a^2+2z_a z_b+z_b^2= x_a^2+y_a^2+z_a^2+x_b^2+y_b^2+z_b^2 \;;$$

$2x_a x_b+2y_a y_b+2z_a z_b=0$; $x_a x_b+y_a y_b+z_a z_b=0$.

This result is equivalent for “Theorem 1” in three-dimensional space. The prove for “Theorem 2” in three-dimensional space is quit symmetric exactly like it was in case of two-dimensional vector space.

Lets generalize definition of orthogonality for vector space of **any dimension.**

For two and three-dimensional vector space, vectors a and b are orthogonal iff $\sum_{i=0}^{N-1} a[i]b[i]=0$.

This result we get from simple geometric laws and Pythagorean Theorem.

For other dimensions, where $N \neq 2$ and $N \neq 3$, the geometric laws are irrelevant.
Therefore, we postulate this **definition**:

Two N-dimensional vectors a and b of real numbers are orthogonal iff $\sum_{i=0}^{N-1} a[i]b[i]=0$.

Where N is integer, $N>0$. The last sum is so important, that deserved for own name: Dot Product of two vectors of real numbers a and b is $(a,b)=\sum_{i=0}^{N-1} a[i]b[i]$. Within Dot Product notation, two vectors of real numbers a and b are orthogonal iff $(a,b)=0$.

Remind definition of vector length (norm). In N-dimensional space **length of vector** a is defined like $|a|=\sqrt{\sum_{n=0}^{N-1} a^2[n]}$ and corresponding square length is $|a|^2=\sum_{n=0}^{N-1} a^2[n]$.
Within Dot Product notation for N-dimensional vector of real numbers a, its square length is $|a|^2=(a,a)$.

Complex Dot Product and Orthogonality.

Dot Product concept can be extended to cover the Space of Complex Vectors.
Lets start from square length of vector. By definition $|a|^2=(a, a)$.
Any complex number I will write here in this form: $a=a_r+ja_i$. Where $j^2=-1$, a_r is called real part and a_i imaginary part of complex number a . It is common to draw complex number as two-dimensional vector. Square length of this vector could be computed like this:
$\|a\|^2=a_r^2+a_i^2$.
For vector of N complex numbers its square length could be computed as square length of 2N-dimensional vector, where listed both – real and imaginary parts of each vector coordinate.
This square length is:

$$\|a\|^2=\sum_{n=0}^{N-1}(a_r^2[n]+a_i^2[n]) =$$

($aa^*=(a_r+ja_i)(a_r-ja_i)=a_r^2+a_i^2$)

$$=\sum_{n=0}^{N-1}a[n]a^*[n] .$$

Where a^* is a complex conjugate of a . Conjugate of complex number a is signed a^* and defined like this: if $a=a_r+ja_i$ then $a^*=a_r-ja_i$.

Orthogonality of **complex** number vectors can be defined by geometry laws only for **one-dimensional** vectors. Given two one-dimensional vectors of complex numbers a and b , $a=a_r+ja_i$ and $b=b_r+jb_i$. Take two points A and B in complex plane, corresponding to the given vectors: $A=(a_r, a_i)$ and $B=(b_r, b_i)$.
If vectors A and B are orthogonal, then $a_r b_r+a_i b_i=0$ according to the “Theorem 1”.

With geometrical interpretation of orthogonality, there is very important, although trivial **law of symmetry**: If a is orthogonal to b, then b is orthogonal to a.

Suppose this **definition** of complex vectors orthogonality:
Given two N-dimensional vectors of complex numbers a and b ,

we say a is orthogonal to b iff $\sum_{n=0}^{N-1} a[n]b^*[n]=0$.

Obviously, b is orthogonal to a iff $\sum_{n=0}^{N-1} b[n]a^*[n]=0$.

For one-dimensional complex vectors a and b , where
$a=\{(a_r+ja_i)\}$ and $b=\{(b_r+jb_i)\}$:

- Orthogonality of a to b mean $(a_r+ja_i)(b_r-jb_i)=0$.
- Orthogonality of b to a mean $(a_r-ja_i)(b_r+jb_i)=0$.

Because of **symmetry law** we get two equations that must be valid simultaneously:

$$\begin{Bmatrix}(a_r+ja_i)(b_r-jb_i)=0\\(a_r-ja_i)(b_r+jb_i)=0\end{Bmatrix} \text{ That can be rewrote: } \begin{Bmatrix}a_r b_r-ja_r b_i+ja_i b_r+a_i b_i=0\\a_r b_r+ja_r b_i-ja_i b_r+a_i b_i=0\end{Bmatrix}$$

Lets sum the first equation with the second by its left and right parts:
$2a_r b_r+2a_i b_i=0$ or $a_r b_r+a_i b_i=0$.
If now we represent complex number as two-dimensional vector of real numbers, we get this immediate result: Orthogonality condition for one-dimensional vectors of complex numbers is equivalent to orthogonality of two-dimensional vectors of real numbers.

Summary of Complex **Vector** Orthogonality.

Natural expansion of orthogonality condition to vector space of Complex numbers is:

$$\sum_{n=0}^{N-1} a[n]b^*[n]=0 \ .$$

From all the above calculation we get natural extent of Dot Product concept to the field

of complex number vectors: $(a,b)=\sum_{n=0}^{N-1} a[n]b^*[n]=0$.

With complex Dot Product notation, we get this two final definitions:

- Two complex vectors a and b are orthogonal iff $(a,b)=0$.
- The square length of complex vector a is $\|a\|^2=(a,a)$.

Orthogonality of Continuous Complex Functions.

Lets take a pair of complex continuous functions φ and ψ. My goal is to define orthogonality of φ and ψ in range between real numbers a and b. For simplicity I will restrict $a<b$, although this restriction is artificial.
Split range $[a,b]$ to N equal pieces. From this pieces of range $[a,b]$ and continuous functions φ and ψ we can define two complex vectors $\underline{\varphi}$ and $\underline{\psi}$, N elements each:

$$\underline{\varphi}[n]=\varphi(a+\frac{b-a}{N}n) \quad \text{for} \quad 0\le n<N$$

$$\underline{\psi}[n]=\psi(a+\frac{b-a}{N}n) \quad \text{for} \quad 0\le n<N$$

Vector $\underline{\varphi}$ is orthogonal to vector $\underline{\psi}$ iff $\sum_{n=0}^{N-1}\underline{\varphi}[n]\underline{\psi}^*[n]=0$. The same condition can be rewritten like this: $\sum_{n=0}^{N-1}\varphi(a+\frac{b-a}{N}n)\psi^*(a+\frac{b-a}{N}n)=0$.

Multiply left and right parts of the equation by $\frac{b-a}{N}$:

$$\frac{b-a}{N}\sum_{n=0}^{N-1}\varphi(a+\frac{b-a}{N}n)\psi^*(a+\frac{b-a}{N}n)=0 .$$

The left part of the last equation, by definition of Definite Integrals and Riemann Sum, if N is taken to its positive infinite limit, it is Definite Integral: $\int_a^b \varphi(x)\psi^*(x)dx$.

From all the above calculations we get definition of
Orthogonality for Complex Continuous Functions:

Two complex continuous functions φ and ψ are **orthogonal** in range $[a,b]$ iff $\int_a^b \varphi(x)\psi^*(x)dx=0$.

Given this formula, we define an extension of concept **Dot Product** for continuous functions. In this case it is referred to as Inner Product and is defined for continuous complex functions φ and ψ:

$$(\varphi,\psi)=\int_a^b \varphi(x)\psi^*(x)dx .$$

With Inner Product notation, similar to Dot Product notation, condition of orthogonality is written like $(\varphi,\psi)=0$.

References:

"Base of Math Analysis" by G.M.Fihtengoltz (Russian edition)

"Base of Math Analysis" by V.A.Ilyin and E.G.Posnyak (Russian edition)

``Mathematics of the Discrete Fourier Transform (DFT), with Music and Audio Applications'', by Julius O. Smith III, W3K Publishing, 2003, ISBN 0-9745607-0-7.

Discrete-Time Speech Signal Processing: Principles and Practice, 1/e
Thomas F. Quatieri
ISBN: 013242942X | Prentice Hall PTR

"The FFT Demystified." by Adrian Hey
Engineering Productivity Tools Ltd. http://www.eptools.com/tn/T0001/INDEX.HTM

"Dot Product." Eric W. Weisstein. From *MathWorld*--A Wolfram Web Resource.
http://mathworld.wolfram.com/DotProduct.html